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The dark soliton on a cnoidal wave background

H J Shin

Department of Physics and Research Institute of Basic Science, Kyung Hee University, Seoul 130-701, Korea

E-mail: hjshin@khu.ac.kr

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Abstract

We find a solution of the dark soliton lying on a cnoidal wave background in a defocusing medium. We use the method of Darboux transformation, which is applied to the cnoidal wave solution of the defocusing nonlinear Schrödinger equation. Interesting characteristics of the dark soliton, i.e., the velocity and greyness, are calculated and compared with those of the dark soliton lying on a continuous wave background. We also calculate the shift of the crest of the cnoidal wave along the soliton.

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1. Introduction

Dark solitons are localized holes on a continuous wave (cw) background. There have been many interesting physical phenomena of dark soliton propagation on a modulationally stable background. These include temporal dark solitons in optical fibres [1], spatial dark solitons in waveguides [2] and high-frequency dark solitons in thin magnetic films [3], to name a few. In particular, optical dark solitons have been investigated in many theoretical and experimental papers; see the review [4, 5] and references therein. Recent experimental achievements such as light-induced structured waveguides give rise to a new interest in the application of optical dark solitons [6–8].

In general, a dark soliton is described by generalized (coupled) nonlinear Schrödinger equations (NLSEs). Various properties of dark solitons of these equations have been discussed including soliton stability, dark gap solitons, solitary waves of nonintegrable models, vector dark solitons and their generalizations by coupled NLSEs [9], and (2 + 1)-dimensional dark solitons having circular symmetry [10]. In some special but important cases, dark solitons are described by the integrable nonlinear Schrödinger equation having a normal group velocity dispersion:

$$\partial_{\bar{z}}\psi = -\mathrm{i}\partial_z^2\psi + 2i|\psi|^2\psi, \tag{1}$$

where $\partial_z = \frac{\partial}{\partial z}$, $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}$, and $\bar{z} \equiv x$ and $z \equiv t - x/v_g$ represent the distance of propagation along the fibre and (retarded) time. The integrability of the equation admits the use of the inverse scattering method, which gives a dark soliton solution of

$$\psi(z, \bar{z}) = p[B\tanh\{pB(z - v\bar{z} + 2pA\bar{z})\} + \mathrm{i}A] \exp(-\mathrm{i}vz/2 + 2\mathrm{i}p^2\bar{z} + \mathrm{i}v^2\bar{z}/4), \tag{2}$$

where parameters A and B are connected by a relation $A^2 + B^2 = 1$. In such a form, the dark soliton solution is characterized by three parameters: p and v describe the amplitude and wave number of the cw background, while the parameter A (or B) characterizes the dark soliton itself (so-called greyness).

Considering the potential applicability of the dark solitons, it is desirable to have a more generalized form of dark soliton solutions having more parameters. This would give more freedom in controlling solitons in a given environment. One possible scheme in this direction is to use a fluctuating cw background instead of the plane cw background used in equation (2). It is well known that integrable nonlinear equations have fluctuating cw solutions called cnoidal waves. The integrability also guarantees the existence of a solution of type 'soliton + cnoidal wave'. In this paper, we construct a dark soliton moving on a cnoidal wave background in a defocusing medium. In fact, there arise new interests on these types of solutions which are needed, for example, in describing localized states in optically induced refractive index gratings [6–8].

To find a solution of the required form, we employ a simple but very powerful soliton finding technique based on the Darboux transformation (DT) [11–13]. This method is essentially equivalent to the inverse scattering method (ISM) but avoids the mathematical technicalities of the ISM. Section 2 introduces a solution of the associated linear equation of NLSE, which is required in the DT method. Some specific feature in applying the DT method to equation (1) is explained for the case of plane cw background in section 3. The plane cw background corresponds to the $k \to 1$ limit of the solution which will be obtained in section 4. Finally, section 4 calculates the dark soliton solution on a cnoidal wave background. Interesting characteristics of the found solution are analysed and compared with those of soliton solutions on a plane cw background in section 5.

2. Sym's solution

The defocusing nonlinear Schrödinger (DNLS) equation (1) describes light propagation in a medium whose group velocity dispersion is normal or where waveguide is self-defocusing. It has the following cnoidal wave solution:

$$\psi_c(z, \bar{z}) = -\mathrm{i}kp\,\mathrm{sn}(\chi + K, k)\,\mathrm{e}^{\mathrm{i}\zeta}, \tag{3}$$

where $\chi = p(z - v\bar{z})$, $\zeta = [-vz/2 + p^2(1 + k^2)\bar{z} + v^2\bar{z}/4]$, K $(K' \equiv K(k'))$ is a complete elliptic integral of the first kind and sn is the standard Jacobi elliptic function. v is the velocity of the cnoidal wave and $k \in (0, 1)$ is the modulus of the Jacobi function. As far as elliptic functions are involved we employ the terminology and notation of [14] without further explanation. To obtain a superposed 'soliton + cnoidal wave' solution using the DT method, we need to first find a solution of the following linear equations associated with the DNLS equation (Lax pair):

$$\begin{aligned}
&(\partial_z + \mathrm{i}\lambda/2)s_1 + \psi_c s_2 = 0, \qquad (\partial_z - \mathrm{i}\lambda/2)s_2 + \psi_c^* s_1 = 0, \\
&(\partial_{\bar{z}} - \mathrm{i}|\psi_c|^2 - \mathrm{i}\lambda^2/2)s_1 - (\mathrm{i}\partial_z\psi_c + \lambda\psi_c)s_2 = 0, \\
&(\partial_{\bar{z}} + \mathrm{i}|\psi_c|^2 + \mathrm{i}\lambda^2/2)s_2 + (\mathrm{i}\partial_z\psi_c^* - \lambda\psi_c^*)s_1 = 0,
\end{aligned} \tag{4}$$

where λ is an arbitrary complex number.

The solution of the linear equation (4) for the cnoidal wave (ψ_c) was first introduced by Sym in a different context (description of vortex motion in hydrodynamics) [15]. It was then applied to NLSE-related problems in [16–18]. A more detailed proof of Sym's solution (in a slightly different notation) is given in appendix A of [18]. Sym's solution in the case of DNLS equation is obtained from that of [15, 16] (case of focusing NLSE) by taking $p \to -\mathrm{i}kp$, $u \to ku$ and $k \to \mathrm{i}k/k'$. Sym's solution for the DNLS case is

$$\begin{aligned} s_2 &= \exp(-\mathrm{i}\zeta/2)\,\exp(\mathrm{i}\gamma\bar{z} + k\beta\chi)\frac{\Theta_s(\chi - u)}{\Theta_s(\chi)}, \\ s_1 &= \mathrm{i}k\,\exp(\mathrm{i}\zeta)\frac{\mathrm{sn}(u,k)\,\mathrm{cn}(\chi - u, k)}{\mathrm{dn}(\chi - u, k)} s_2. \end{aligned} \tag{5}$$

The parameter λ is given by

$$\lambda = v/2 - \mathrm{i}p\frac{\mathrm{dn}(u,k)\,\mathrm{cn}(u,k)}{\mathrm{sn}(u,k)}, \tag{6}$$

and γ, β in equation (5) are

$$\begin{aligned} \gamma &= \frac{p^2}{2}\left[k^2\,\mathrm{cn}^2(u,k) + \frac{\mathrm{dn}^2(u,k)}{\mathrm{sn}^2(u,k)}\right], \\ \beta &= \frac{\Theta_s'(u)}{k\Theta_s(u)} + \frac{1}{2}\frac{\mathrm{dn}(u,k)\,\mathrm{cn}(u,k)}{k\,\mathrm{sn}(u,k)} + \frac{k\,\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)}{\mathrm{dn}(u,k)}. \end{aligned} \tag{7}$$

Here,

$$\Theta_s(u) = \theta_4\left(\frac{\mathrm{i}\pi u}{2K'}\right) = 1 + 2\sum(-)^n q^{n^2}\cos\left(\frac{\mathrm{i}n\pi u}{K'}\right), \tag{8}$$

with $q = -\exp(-\pi K/K')$.

3. The dark soliton on a plane wave background

A new solution describing a superposed state of 'soliton + cnoidal wave' is constructed using the DT method [11–13] as follows:

$$\psi_{c-s}(z,\bar{z}) = \psi_c(z,\bar{z}) + 2\,\mathrm{Im}\,\lambda\left(\frac{s_1^*}{s_2^*} - \frac{s_2}{s_1} - \frac{\epsilon N}{s_1 s_2^*}\right)^{-1}, \tag{9}$$

where N is an arbitrary constant and ϵ is a parameter which will be taken as a zero limit value at the end. Equation (9) is the result of the standard DT except for an auxiliary term ϵN, which is needed in the case of a defocusing medium. Using that s_i satisfy the associated linear equations in equation (4), it can be explicitly checked that ψ_{c-s} in equation (9) is a new solution of the DNLS equation.

To explain the DT procedure as well as the role of ϵN term in equation (9), we first treat the case of the plane wave background. The planes wave is obtained from the cnoidal wave in equation (3) by taking the limit of $k \to 1$, which gives $\psi_{\mathrm{pw}} = -\mathrm{i}p\,\exp(\mathrm{i}\zeta_{\mathrm{pw}}) = -\mathrm{i}p\,\exp(-\mathrm{i}vz/2 + 2\mathrm{i}p^2\bar{z} + \mathrm{i}v^2\bar{z}/4)$. Using the fact that for $k \to 1$, $K \to \infty$, $K' \to \pi/2$, $\Theta_s \to 1$, we obtain s_1, s_2 in the plane wave limit as

$$s_2 = \exp[-\mathrm{i}\zeta_{\mathrm{pw}}/2 + \mathrm{i}\gamma_{\mathrm{pw}}\bar{z} + \beta_{\mathrm{pw}}p(z - v\bar{z})], \qquad s_1 = \mathrm{i}\,\exp(\mathrm{i}\zeta_{\mathrm{pw}})\,(\tanh u)s_2, \tag{10}$$

where u is a complex parameter related to λ as

$$\lambda = \frac{v}{2} - \mathrm{i}p\,\mathrm{sech}\,u\,\mathrm{csch}\,u, \tag{11}$$

and

$$\gamma_{\mathrm{pw}} = \frac{p^2}{2}(\mathrm{sech}^2 u + \mathrm{csch}^2 u), \qquad \beta_{\mathrm{pw}} = \frac{1}{2}\mathrm{sech}\, u\, \mathrm{csch}\, u + \tanh u. \tag{12}$$

Then, the application of DT in equation (9) with $\epsilon \to 0$ gives

$$\psi_{c-s} = -\mathrm{i}p\frac{\mathrm{i}\sin(2\,\mathrm{Im}\, u) + \sinh(2\,\mathrm{Re}\, u)}{\mathrm{i}\sin(2\,\mathrm{Im}\, u) - \sinh(2\,\mathrm{Re}\, u)}\, \exp(\mathrm{i}\zeta_{\mathrm{pw}}). \tag{13}$$

Equation (13) is again a plane wave but with a shifted phase and is not the desired type of 'dark soliton + plane wave'.

To obtain a dark soliton solution, we need to take a special value on $\mathrm{Im}\, u$ such that $u = w - \mathrm{i}\pi/4 + \mathrm{i}\epsilon$ with an arbitrary real w. In this case with the limit of $\epsilon \to 0$, the DT in equation (9) with $N = 4\,\mathrm{sech}\, 2w$ becomes the well known 'dark soliton + plane wave' solution in equation (2) with $A = \mathrm{sech}\, 2w$, $B = \tanh 2w$. We note that (with $u = w - \mathrm{i}\pi/4 + \mathrm{i}\epsilon$)

$$\begin{aligned} \mathrm{Im}\,\lambda &= -8\epsilon p\frac{\sinh(2w)}{\cosh(4w) + 1} + O(\epsilon^2), \\ |s_1|^2 - |s_2|^2 &= -4\epsilon\, \mathrm{sech}(2w)\, \exp\{2\beta_{\mathrm{pw}} p(z - v\bar{z})\} + O(\epsilon^2). \end{aligned} \tag{14}$$

Thus, the three terms in equation (9), i.e., $\mathrm{Im}\,\lambda$, $s_1^*/s_2^* - s_2/s_1$ and ϵN, are of the order of $O(\epsilon^1)$ and gives the dark soliton solution under the limit of $\epsilon \to 0$. This limiting procedure can be explained more clearly from the viewpoint of vector NLSE; see the appendix.

4. The dark soliton on a cnoidal wave background

As in the case of plane wave background, a new solution using equations (9) and (5) with an arbitrary complex parameter u is not the type of 'soliton + cnoidal wave', but is just a cnoidal wave shifted along z. To obtain a new solution of the 'soliton + cnoidal wave' type, we need to take a special value on $\mathrm{Im}\, u$ as

$$u = w + \mathrm{i}\frac{K'}{2} + \mathrm{i}\frac{\epsilon}{k}, \tag{15}$$

where w is an arbitrary real parameter, ϵ is the parameter appearing in equation (9) and $K' = K(k')$. For this value of u, $\mathrm{Im}\,\lambda$ in equation (6) becomes

$$\mathrm{Im}\,\lambda = 4p\epsilon(1 + k)\frac{\mathrm{sn}\, w\, \mathrm{dn}\, w\, \mathrm{cn}\, w}{(1 + k\, \mathrm{sn}^2 w)^2} + o(\epsilon^2). \tag{16}$$

Similarly, s_1/s_2 in equation (5) for the value of u in equation (15) becomes

$$\begin{aligned} s_1/s_2 = -\exp(\mathrm{i}\zeta)&\left(1 + 2\epsilon\frac{1 - k^2\, \mathrm{sn}^2 w - k^2\, \mathrm{sn}^2 Z + k^2\, \mathrm{sn}^2 w\, \mathrm{sn}^2 Z}{k(1 + k\, \mathrm{sn}^2 w)(1 - k\, \mathrm{sn}^2 Z)} + O(\epsilon^2)\right) \\ &\times \frac{\{(1 + k)\, \mathrm{sn}\, w + \mathrm{i}\, \mathrm{cn}\, w\, \mathrm{dn}\, w\}(\mathrm{cn}\, Z + \mathrm{i}\, \mathrm{sn}\, Z\, \mathrm{dn}\, Z)}{(1 + k\, \mathrm{sn}^2 w)(\mathrm{i dn}\, Z - k\, \mathrm{sn}\, Z\, \mathrm{cn}\, Z)}, \end{aligned} \tag{17}$$

where $Z = \chi - w = p(z - v\bar{z}) - w$. Note that

$$|s_1/s_2|^2 = 1 + 4\epsilon\frac{1 - k^2\, \mathrm{sn}^2 w - k^2\, \mathrm{sn}^2 Z + k^2\, \mathrm{sn}^2 w\, \mathrm{sn}^2 Z}{k(1 + k\, \mathrm{sn}^2 w)(1 - k\, \mathrm{sn}^2 Z)} + O(\epsilon^2), \tag{18}$$

such that $\mathrm{Im}\,\lambda$, $s_1^*/s_2^* - s_2/s_1$ and ϵN are of the order of $O(\epsilon^1)$. This property is required to obtain the 'soliton + cnoidal wave' solution under the limiting procedure $\epsilon \to 0$; see section 3 and the appendix.

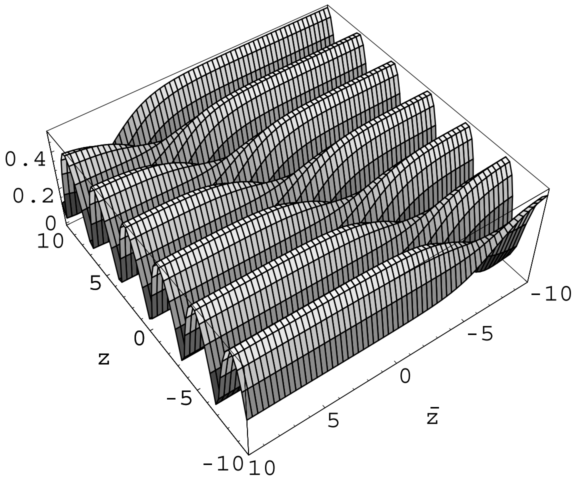

Figure 1. $|\psi_{c-s}|$ shows a dark soliton residing on a cnoidal wave background. The parameters are $v = 0, k = 0.5, p = -1, w = 1, M = 1$.

Applying these results to equation (9), we can obtain the 'soliton + cnoidal wave' solution

$$\psi_{c-s} = -\mathrm{i}kp\left\{\frac{\mathrm{sn}(Z+K-w)\,\exp(\mathrm{i}\delta)+\mathrm{sn}(Z+K+w)}{2}\right.$$
$$\left. -\frac{\mathrm{sn}(Z+K-w)\,\exp(\mathrm{i}\delta)-\mathrm{sn}(Z+K+w)}{2}\tanh D\right\}\exp(\mathrm{i}\zeta), \tag{19}$$

where

$$D = \mathrm{Im}\,\gamma\bar{z} - k\,\mathrm{Re}\,\beta\chi + \frac{1}{2}\ln\left(M\left|\frac{\Theta_s(\chi)}{\Theta_s(Z-\mathrm{i}K'/2)}\right|^2\frac{1-k\,\mathrm{sn}^2 Z}{\mathrm{dn}^2 w - k^2\,\mathrm{cn}^2 w\,\mathrm{sn}^2 Z}\right), \tag{20}$$

$$M = \frac{(1+k\,\mathrm{sn}^2 w)N}{4},$$
$$\mathrm{Im}\,\gamma = -2kp^2\frac{(1+k)\,\mathrm{sn}\,w\,\mathrm{cn}\,w\,\mathrm{dn}\,w}{(1+k\,\mathrm{sn}^2 w)^2}, \tag{21}$$
$$\mathrm{Re}\,\beta = \mathrm{Re}\frac{\Theta_s'(w+\mathrm{i}K'/2)}{k\Theta_s(w+\mathrm{i}K'/2)} + 2\frac{\mathrm{sn}\,w\,\mathrm{cn}\,w\,\mathrm{dn}\,w}{1-k^2\,\mathrm{sn}^4 w},$$

and

$$\exp(\mathrm{i}\delta) = \frac{1-2(1+k+k^2)\,\mathrm{sn}^2 w + k^2\,\mathrm{sn}^4 w - 2\mathrm{i}(1+k)\,\mathrm{sn}\,w\,\mathrm{dn}\,w\,\mathrm{cn}\,w}{(1+k\,\mathrm{sn}^2 w)^2}. \tag{22}$$

Equation (19) is the main result of this paper, describing the dark soliton moving on a cnoidal wave background.

5. Some characteristics of the solution

Figure 1 shows $|\psi_{c-s}|$, obtained by using equations (19) and (20). It shows the characteristic dark soliton of DNLS equation where the dark soliton resides on a cnoidal wave background. The parameters used for figure 1 are $v = 0, k = 0.5, p = -1, w = 1, M = 1$. This figure is

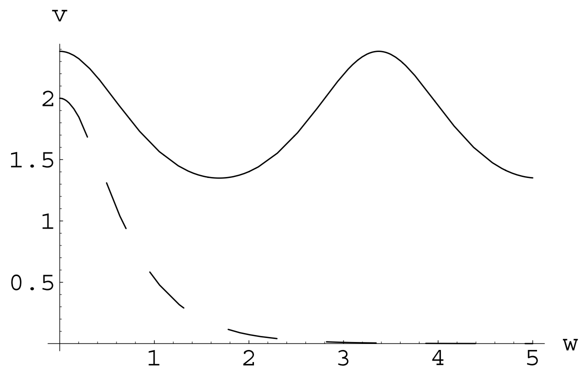

Figure 2. Velocity v_s^0 versus w of a dark soliton for $k = 0.5$ (solid line) and $k = 1$ (dashed line). The parameter is $p = -1$.

drawn using MATHEMATICA, which is also used to check that the solution in equation (19) indeed satisfies the DNLS equation (1).

The dark soliton moves along a line given by $D = 0$ in equation (20). In determining the moving direction of the soliton, we need a careful treatment of the second term in D of equation (20). Due to the following quasi-periodicity of the theta function,

$$\Theta_s(z + 2K) = \exp(\pi K/K' + \pi z/K')\Theta_s(z), \tag{23}$$

the last term in D can be written in two parts: the first one, which is proportional to χ as $\pi w \chi/(2KK')$ and the second one, which is a periodic function in χ. The first one contributes to the determination of the moving direction of the soliton, while the second one gives a wiggling behaviour of the dark soliton along its moving direction. Considering this fact, the moving direction (from $D = 0$), or the velocity $v_s \equiv z/\bar{z}$ of the dark soliton, is given by $v_s = v + v_s^0$ with (v is the velocity of the cnoidal wave in equation (3))

$$v_s^0 = -2kp \frac{(1+k)\,\mathrm{sn}\,w\,\mathrm{cn}\,w\,\mathrm{dn}\,w}{(1+k\,\mathrm{sn}^2 w)^2} \left(\mathrm{Re} \frac{\Theta_s'(w + \mathrm{i}K'/2)}{\Theta_s(w + \mathrm{i}K'/2)} + \frac{2k\,\mathrm{sn}\,w\,\mathrm{cn} w\,\mathrm{dn}\,w}{1 - k^2\,\mathrm{sn}^4 w} - \frac{\pi w}{2KK'} \right)^{-1}. \tag{24}$$

Note that v_s^0 is a periodic function in w with the periodicity $2K$. Figure 2 shows the velocity v_s^0 in w for a cnoidal wave background ($k = 0.5$, solid line) and for a plane wave background ($k = 1$, dotted line). It shows that a soliton lying on a cnoidal wave background moves faster than that lying on a plane wave background. This trend is also seen in figure 3, where we plot v_s^0 in k for $w = 1, 2, 3$. From equation (24), we can find that the fastest velocity $v_{\max}$ for a given k (in the case of $p < 0$) is given at $w = 0, 2K, 4K, \ldots$, with

$$v_{\max} = -2p \frac{(1+k)k}{1 + k - E/K}. \tag{25}$$

In particular, at $k = 0$ and $k = 1$, $v_{\max} = -2p$. Using numerical calculation, we found that $v_{\max}$ itself attains its largest value $-2.48p$ (in the case of $p < 0$) at $k = 0.82$. On the other hand, the slowest velocity $v_{\min}$ for a given k (in the case of $p < 0$) is given at $w = K, 3K, \ldots$, which is

$$v_{\min} = -2p \frac{(1-k)k}{-1 + k + E/K}. \tag{26}$$

Other special values are $v_s^0 = -2p$ at $k = 0$ and $v_s^0 = -2p\,\mathrm{sech}\,2w$ at $k = 1$ (plane wave).

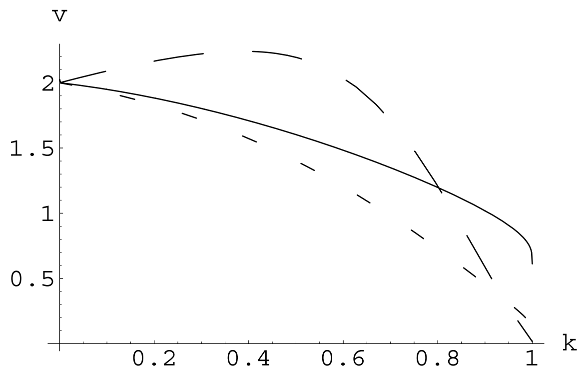

Figure 3. Velocity v_s^0 versus k of a dark soliton for $w = 1$ (solid line), $w = 2$ (dotted line), $w = 3$ (dashed line). $k = 1$ corresponds to the plane wave background. The parameter is $p = -1$.

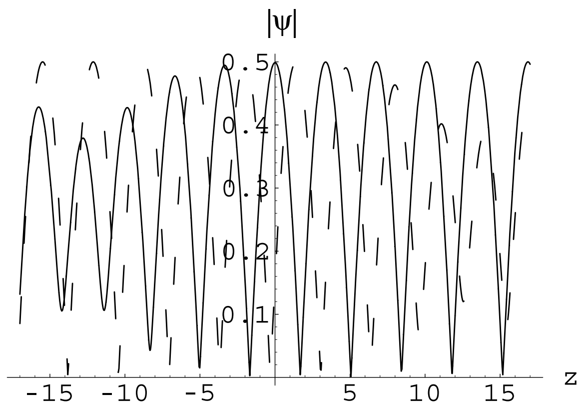

Figure 4. $|\psi|$ at $\bar{z} = -8$ (solid line) and $\bar{z} = 8$ (dotted line) shows a dark soliton. They show the shift of crest at a region around $z = 0$. The parameters are $v = 0$, $k = 0.5$, $p = -1$, $w = 1$, $M = 1$.

Another interesting feature of figure 1 is that the crest of the cnoidal wave shifts constantly across the dark soliton. As we move away from the soliton such that $D \to -\infty$ in equation (19), we have $\psi_{c-s} \to -\mathrm{i}kp\,\mathrm{sn}(Z + K - w)\exp(\mathrm{i}\zeta + \mathrm{i}\delta)$. Similarly, $\psi_{c-s} \to -\mathrm{i}kp\,\mathrm{sn}(Z + K + w)\exp(\mathrm{i}\zeta)$ for $D \to \infty$. Thus the shift of crests is $2w$. Figure 4 plots $|\psi|$ at two values of $\bar{z} = -8$ (solid line) and 8 (dotted line) using the same parameters used in figure 1. The centre of the dark soliton for $\bar{z} = -8$ lies around $z = -13$ while it lies around $z = 13$ for $\bar{z} = 8$. The shift in figure 4 is clearly seen for $-5 < z < 5$, which is $2w = 2$.

During this shift, a dark soliton appears. The intensity of the dark soliton, known as the greyness, can be seen by studying the coefficient of $\tanh D$ in equation (19),

$$C \equiv \left| \frac{\mathrm{sn}(Z + K - w)\exp(\mathrm{i}\delta) - \mathrm{sn}(Z + K + w)}{2} \right|$$
$$= \left| \frac{(1 + k)\,\mathrm{sn}\,w\,\mathrm{dn}\,w\,\mathrm{cn}\,w}{1 + k\,\mathrm{sn}^2 w} \frac{1 - k\,\mathrm{sn}^2 Z}{\mathrm{dn}^2 w - k^2\,\mathrm{cn}^2 w\,\mathrm{sn}^2 Z} \right|. \quad (27)$$

In particular, $C = 0$ for $w = 0, K$ for any k. At $w = K/2$,

$$C = \frac{\sqrt{2}(1+k+k'+kk')^{3/2}}{(1+k+k')^2}\frac{1-k\,\mathrm{sn}^2 Z}{1+k'-k^2\,\mathrm{sn}^2 Z}. \tag{28}$$

The largest value of C is obtained when $\mathrm{sn}\, Z = 0$, which is

$$C_{\max} = \frac{\sqrt{2}\sqrt{1+k'}(1+k)^{3/2}}{(1+k+k')^2}. \tag{29}$$

In particular at $k = 1$ (plane wave background), $C_{\max} = 1$ which means a true dark soliton can arise. Generally at $k \neq 1$, $C_{\max} < 1$ and only grey solitons are arisen on cnoidal wave backgrounds. The cnoidal wave background makes a dark soliton more grey than the plane wave background.

6. Conclusion

In this paper, we have introduced a 'soliton + cnoidal wave' solution of the DNLS equation. It was obtained using the DT method and Sym's solution of the associated linear equation on a cnoidal wave background. We calculate the moving direction of a soliton on a cnoidal wave background and the shift of the crest of a cnoidal wave. We also discuss the greyness of the soliton on a background. These types of solutions, though they can be easily applicable to the analysis of physically interesting processes, seem rather rare in the literature of physics. The stability analysis of these solutions remains for future study. In fact, there appear some numerical studies on this subject (vector NLSE of the defocusing case) [8].

Acknowledgment

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Appendix. Explanation of equation (9)

In this appendix, we explain the $\epsilon \to 0$ procedure in sections 3 and 4 from the viewpoint of the vector NLSE:

$$\partial_{\bar{z}}\psi_i = -\mathrm{i}\partial_z^2\psi_i + 2\mathrm{i}(|\psi_1|^2 + |\psi_2|^2)\psi_i, \qquad i = 1, 2. \tag{A.1}$$

The linear equations to be solved in this case are those of equation (4) augmented with

$$(\partial_z - \mathrm{i}\lambda/2)s_3 = 0, \qquad (\partial_{\bar{z}} + \mathrm{i}\lambda^2/2)s_3 = 0. \tag{A.2}$$

In this case, the DT formula giving the 'soliton + cnoidal wave' solution is [19, 18]

$$\psi_{1,c-s}(z,\bar{z}) = \psi_c(z,\bar{z}) + 2\,\mathrm{Im}\,\lambda\frac{s_1 s_2^*}{|s_1|^2 - |s_2|^2 - |s_3|^2},$$
$$\psi_{2,c-s}(z,\bar{z}) = 2\,\mathrm{Im}\,\lambda\frac{s_1 s_3^*}{|s_1|^2 - |s_2|^2 - |s_3|^2}. \tag{A.3}$$

Note that the DT formula (A.3) works for any solution $s_i, i = 1, 3$, and for any complex-valued λ. When we take $s_3 = 0$, we obtain a solution for the single-component NLSE (i.e., $\psi_{2,c-s} = 0$), but just with a shifted phase of ψ_c, as explained in section 3. Instead, we take $s_3 = \sqrt{\epsilon N}\exp(\mathrm{i}\lambda z/2 - \mathrm{i}\lambda^2\bar{z}/2)$ and take $\mathrm{Im}\,\lambda$, $|s_1|^2 - |s_2|^2$ are of the order of $O(\epsilon^1)$, while s_1, s_2 are of the order of $O(\epsilon^0)$. Then, the DT formula (A.3) becomes equation (9), which gives the solutions in sections 3 and 4.

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